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AN AUTOMORPHISM OF ORTHOGONAL ALGEBRAIC K-THEORY

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One of the themes of higher algebraic K-theory is that there is often a connection between apparently unrelated topics in algebra and topology. Such is particularly the case between matrix groups over finite fields and certain topological spaces related to classification of vector bundles and spherical fibrations. This was first realized by Quillen [10, 11] and was systematically explored in our monograph [6]. In this paper we continue in this vein by establishing a connection between the process of changing a quadratic form on a vector space over a finite field and a certain map $\Delta : \mathrm{BO} \to \mathrm{SO}$ related to Bott periodicity.

To be more precise: Let V be a finite dimensional vector space over $\mathbb{F}_q$ (q odd) with non-degenerate quadratic form Q. Let $O(V, Q)$ denote the corresponding orthogonal group. Let $\mu \in \mathbb{F}_q$ be a nonsquare. Then $O(V, Q)$ can be regarded as operating on $(V, \mu Q)$ thus inducing a homomorphism $O(V, Q) \to O(V, \mu Q)$. This process defines an infinite loop map $\Phi : \Gamma \to \Gamma$ on the infinite loop space Γ obtained from the orthogonal groups over $\mathbb{F}_q$ (see Section 1). This map Φ was instrumental in [6] in the study of the homology of the finite orthogonal groups and homology operations in Γ. In fact, it is the existence of this automorphism which accounts for the essential differences between the orthogonal algebraic K-theory of $\mathbb{F}_q$ and the ordinary algebraic K-theory of $\mathbb{F}_q$ studied by Quillen [11].

In view of the close parallels between algebra and topology already established in this situation, one would expect that the map Φ, arising in this algebraic context, should have some interesting geometric interpretation. Moreover, the need to find such a geometric interpretation is not merely a matter of esthetics: although one knows the homotopy groups of Γ, one can not easily determine the action of Φ on $\pi_*\Gamma$. The point is that Φ is defined algebraically and is transferred to the geometric context by means of group completion which behaves very badly on homotopy groups.

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The main result of this paper provides such a purely geometric interpretation of the map $\Phi : \Gamma \to \Gamma$. In [6; III Th.3.1d] we showed that Γ is equivalent as an infinite loop space to JO(q), the homotopy fiber of

$$\mathrm{BO} \xrightarrow{\psi^q - 1} \mathrm{BSO}.$$

Let τ, β, Δ be defined by the fibration sequences

$$\mathrm{SO} \xrightarrow{\tau} \mathrm{JO}(q) \xrightarrow{\beta} \mathrm{BO} \xrightarrow{\psi^q - 1} \mathrm{BSO}, \tag{1}$$

$$\Omega\mathrm{SO} = O/U \longrightarrow \mathrm{BU} \xrightarrow{\varrho} \mathrm{BO} \xrightarrow{\Delta} \mathrm{SO}, \tag{2}$$

where ϱ is realification. Localizing away from $p, \psi^q - 1$ and hence β, τ are infinite loop maps. On the other hand it is well known that (2) is part of a periodic infinite loop space fibration involved in Bott periodicity. (The map Δ has various other interpretations. It can also be described as the adjoint of

$$S' \wedge \mathrm{BO} \xrightarrow{\eta \wedge 1} \mathrm{BO} \wedge \mathrm{BO} \xrightarrow{\otimes} \mathrm{BO}$$

where η is the generator of $\pi_1 \mathrm{BO} \approx \pi_1^s$. Thus Δ is often called η or simply the Bott map.)

Theorem A. *As an infinite loop map,*

$$\Phi \simeq 1 + \tau \circ \Delta \circ \beta.$$

Our second result relates the action of Φ on JO(q) to group representations invariant under ψ^q. Let G be a finite group and let RO(G), $R(G)$ denote the real and complex representation rings of G. Quillen [10] has shown that

$$\hat{R}(G)^{\psi^q} \approx [\mathrm{BU}, \mathrm{JU}(q)]$$

where JU(q) is the fibre of $\psi^q - 1 : \mathrm{BU} \to \mathrm{BU}$. The real case is more delicate.

Theorem B. $\hat{\mathrm{RO}}(G)^{\psi^q} = [\mathrm{BG}, \mathrm{JO}(q)]_\Phi$ *if some odd power of ψ^q acts idempotently on* RO(G).

Here we are using superscripts to denote invariants and subscripts to denote coinvariants. The hypothesis of Theorem B is satisfied if the exponent of G divides $q^s - 1$ for some odd s. We shall show that some hypothesis on ψ^q is necessary in Theorem B, by exhibiting a dihedral group for which the conclusion of Theorem B does not hold.

As corollaries of Theorem A we have:

Corollary C. $\Phi^2 \simeq 1$, $2\Phi \simeq 2$ *as infinite loop space maps.*

Corollary D. *$\Phi_*: \pi_n(\mathrm{JO}(q)) \to \pi_n(\mathrm{JO}(q))$ is the identity except for $n = 8k+1$, in which case*

$$\Phi_*(\eta_k) = \eta_k, \qquad \Phi_*(\mu_k) = \mu_k + \eta_k,$$

where $\pi_{8k+1}(\mathrm{JO}(q)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is generated by η_k, μ_k where $\eta_k \in \operatorname{im} \tau_* = \mathbb{Z}/2$ and $\beta_*(\mu_k)$ generates $\pi_{8k+1}\mathrm{BO} = \mathbb{Z}/2$.

In Section 1 we recall the definition of Φ and prove Theorem A; Corollaries C and D follow easily. Section 2 is devoted to the proof of Theorem B.

All homology groups are taken with coefficients in $\mathbb{Z}/2$ unless expressly stated to the contrary.

1. Definition of Φ and the proof of Theorem A

Before giving the proof of Theorem A, we briefly recall the group theoretic definition of Φ from [6; II 3.12, 4.13]. This is equivalent to but more concrete than the definition given in the Introduction. Let $\mathrm{O}_n(\mathbb{F}_q)$ denote the group of orthogonal $n \times n$ matrices over $\mathbb{F}_q$, i.e. matrices A over $\mathbb{F}_q$ satisfying $AA^t = I_n$. The classifying spaces $\mathrm{BO}_n(\mathbb{F}_q)$ form a monoid

$$M = \coprod_{n \geq 0} \mathrm{BO}_n(\mathbb{F}_q)$$

under Whitney sum

$$\mathrm{BO}_n(\mathbb{F}_q) \times \mathrm{BO}_m(\mathbb{F}_q) \to \mathrm{BO}_{n+m}(\mathbb{F}_q).$$

Moreover, $\Omega\mathrm{BM}$ is an infinite loop space whose zero component we denoted by $\Gamma = \Gamma_0 \mathrm{B}\mathcal{O}(\mathbb{F}_q)$ in [6; II 3.1]. It is on Γ that we shall first define $\Phi: \Gamma \to \Gamma$. To this end choose $a, b \in \mathbb{F}_q$ such that $a^2 + b^2$ is not a square (in the topologically significant case $q \equiv \pm 3 \bmod 8$, $a = b = 1$ is a suitable choice since 2 is not a square). Let $C = \left(\begin{smallmatrix} -a & b \\ b & a \end{smallmatrix}\right)$. $C_n = \bigoplus_{i=1}^n C$ and define the outer automorphism

$$\phi_n : \mathrm{O}_{2n}(\mathbb{F}_q) \to \mathrm{O}_{2n}(\mathbb{F}_q)$$

by

$$\phi_n(A) = C_n A C_n^{-1} \quad \text{for } A \in \mathrm{O}_{2n}(\mathbb{F}_q).$$

Clearly $\phi_{n+m}(A \oplus B) = \phi_n(A) \oplus \phi_m(B)$ for $A \in \mathrm{O}_{2n}(\mathbb{F}_q)$, $B \in \mathrm{O}_{2m}(\mathbb{F}_q)$. In fact, it follows from May's theory of infinite loop spaces [8] that the resulting maps $B\phi_n : \mathrm{BO}_{2n}(\mathbb{F}_q) \to \mathrm{BO}_{2n}(\mathbb{F}_q)$ can be assembled to give an infinite loop map

$$\Phi : \Gamma \to \Gamma.$$

In [6; III 3.1d] we have constructed an equivalence of infinite loop spaces

$$\lambda : \Gamma \xrightarrow{\simeq} \mathrm{JO}(q).$$

By an abuse of notation we shall also denote by

$$\Phi : \mathrm{JO}(q) \to \mathrm{JO}(q)$$

the composite infinite loop map $\lambda\Phi\lambda^{-1}$.

Corollary D asserts that on homotopy groups $\Phi - 1$ is usually zero. On homology groups, however $\Phi - 1$ is highly nontrivial as we shall now. According to [6; I 7.1] the homology of $\mathrm{JO}(q)$ can be computed from the fibration

$$\mathrm{SO} \xrightarrow{\tau} \mathrm{JO}(q) \xrightarrow{\beta} \mathrm{BO}$$

as

$$H_*(\mathrm{JO}(q)) = \mathbb{Z}/2[\bar{v}_1, \bar{v}_2, \ldots] \otimes E[\bar{u}_1, \bar{u}_2, \ldots]$$

where $\bar{u}_k = \tau_*(u_k)$, $\beta_*(\bar{v}_k) = v_k$, and

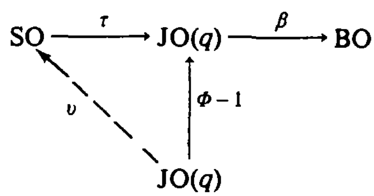
$$H_*(\mathrm{SO}) = E[u_1, u_2, \ldots], \qquad H_*(\mathrm{BO}) = \mathbb{Z}/2[v_1, v_2, \ldots].$$

In [6; IV 3.2] we computed

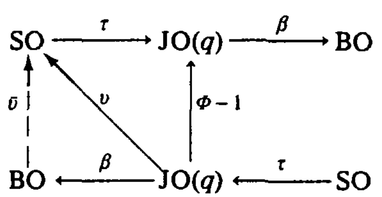
$$(\Phi - 1)_*(\bar{u}_k) = 0, \qquad (\Phi - 1)_*(\bar{v}_k) = \bar{u}_k.$$

We now turn to the proof of Theorem A which occupies the remainder of this section.

In the diagram



$\beta(\Phi - 1) \simeq 0$ as infinite loop maps since β factors through Brauer lifting [6; III Th. 3.1d] and $\Phi \simeq 1$ after passing to the extension $\mathbb{F}_q(\sqrt{a^2 + b^2})$. This follows because conjugation by C is equivalent to conjugation by $(\sqrt{a^2 + b^2})^{-1} C \in O_2(\mathbb{F}_q(\sqrt{a^2 + b^2}))$. Thus there exists an infinite loop map υ making the diagram commute. In the diagram



(1.1)

we wish to show that an infinite loop map $\bar{\upsilon}$ exists and makes the diagram commute. This follows from:

Lemma 1.2. *$\upsilon\tau \simeq 0$ as infinite loop maps.*

Proof. We first observe that $2\upsilon \simeq 0$ since

$$\mathrm{KO}^{-1}(\mathrm{JO}(q)) \approx \mathrm{KO}^{-1}(\mathrm{BO}(\mathbb{F}_q)) \approx \varprojlim \mathrm{KO}^{-1}(\mathrm{BO}_n(\mathbb{F}_q))$$

and $\mathrm{KO}^{-1}(\mathrm{BG})$ is a $\mathbb{Z}/2$-module for any finite group G [4]. Thus

$$(\upsilon\tau)_* = 0 : \pi_*(\mathrm{SO}) \otimes \mathbb{Q} \to \pi_*(\mathrm{SO}) \otimes \mathbb{Q}$$

and if $f \in [\mathrm{BSO}, \mathrm{BSO}]$ is a delooping of $\upsilon\tau$, i.e. $\Omega f = \upsilon\tau$, then

$$f_* = 0 : \pi_*(\mathrm{BSO}) \otimes \mathbb{Q} \to \pi_*(\mathrm{BSO}) \otimes \mathbb{Q}.$$

However, Adams has shown that H-maps $\mathrm{BSO} \to \mathrm{BSO}$ are determined up to homotopy by their induced maps on rational homotopy groups [9; V 2.8] and so $f \simeq 0$. It remains to show $f \simeq 0$ as an infinite loop map, but this follows directly from the Madsen–Snaith–Tornehave result [7; 3.11] that two infinite loop maps $\mathrm{BSO} \rightrightarrows \mathrm{BSO}$ are homotopic (localized at any given prime l) if they are homotopic as ordinary maps, i.e.

$$i : [\mathrm{bso}, \mathrm{bso}] \hookrightarrow [\mathrm{BSO}, \mathrm{BSO}]$$

where bso denotes the connective Ω spectrum whose 0-th space is BSO and i is induced by restriction to the 0-th spaces. □

We now wish to show that we can complete the diagram

$$\begin{array}{ccccc} \mathrm{BO} & \xrightarrow{\Delta} & \mathrm{SO} & \xrightarrow{c} & \mathrm{SU} \\ & {\scriptstyle g}\nwarrow & \uparrow{\scriptstyle \bar{\upsilon}} & & \\ & & \mathrm{BO} & & \end{array} \tag{1.3}$$

with an infinite loop map g. Here c is complexification and so Δ, c is a fibre sequence of infinite loop maps. Let bo, su denote the connected Ω spectra whose 0-th spaces are BO and SU respectively. Then the existence of g follows from:

Proposition 1.4 $[\mathrm{bo}, \mathrm{su}] = 0$.

Lemma 1.5. $[\mathrm{BO}[n, \infty], \mathrm{SU}] = 0$.

Proof. For $n = 1, 2, 3, 4$, $K^{-1}(\mathrm{BO}[n, \infty]) = 0$ by Atiyah–Segal [4], since $\mathrm{BO}[1, \infty] = \mathrm{BO}$, $\mathrm{BO}[2, \infty] = \mathrm{BSO}$, $\mathrm{BO}[4, \infty] = \mathrm{BSpin}$. Following Anderson–Hodgkin [3] the remaining cases are proved by induction using the standard fibrations

$$K(C, n-1) \to \mathrm{BO}[n+1, \infty] \to \mathrm{BO}[n, \infty]$$

where $C = \mathbb{Z}/2$ or $\mathbb{Z}_{(2)}$ according to n. In detail,

$$K^*((K(\mathbb{Z}/2, n)) = 0, \quad n \geq 2 \text{ by [3; Th. 1]},$$

$$K^*(K(\mathbb{Z}_{(2)}, n)) = 0, \qquad n \geq 3 \text{ by [7; 3.2]},$$

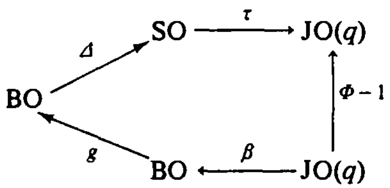
and so $K^*\mathrm{BO}[n+1, \infty] \approx K^*\mathrm{BO}[n, \infty]$ for $n \geq 4$. □

Proof of Proposition 1.4. Anderson [2; Th. 1] also proves that every stable cohomology operation (between the connected forms of complex, real, or symplectic K-theories defined on complexes) is represented by a map of spectra which is unique up to homotopy. Thus

$$[\mathrm{bo}, \mathrm{su}] = \varprojlim\, [\mathrm{BO}[n, \infty], \mathrm{SU}[n, \infty]]. \tag{1.6}$$

By connectivity $[\mathrm{BO}[n,\infty], \mathrm{SU}] = [\mathrm{BO}[n,\infty], \mathrm{SU}[n,\infty]]$ and so by Lemma 1.5 $[\mathrm{bo}, \mathrm{su}] = 0$. □

We can combine diagrams (1.1) and (1.3) to obtain



(1.7)

Lemma 1.8. *$g\beta$ is a homotopy fibre of $\psi^q - 1$ as infinite loop space maps.*

Proof. First we observe that g is an equivalence. In homology

$$\tau_* \Delta_* g_* \beta_*(\bar{v}_k) = (\Phi - 1)_*(\bar{v}_k),$$

$$\tau_* \Delta_* g_*(v_k) = \bar{u}_k,$$

$$\Delta_* g_*(v_k) = u_k,$$

since τ_* is injective. It is well known that $\Delta_*(v_k) = u_k$, hence

$$g_*(v_k) = v_k + \text{decomposables}$$

and g_* is an isomorphism of homology algebras.

Now we use the completion (at 2) of Bousfield–Kan [5]

$$X \to \hat{X}$$

for simple spaces X (see also May [9]). We are forced to consider completions in order to circumvent intractable $\varprojlim^1$ problems, e.g. if Y^α denotes the finite subcomplexes of Y then

$$[Y, \hat{X}] \approx \varprojlim\, [Y^\alpha, \hat{X}]$$

because $\varprojlim^1 = 0$ in the Milnor sequence

$$0 \to \varprojlim^1 [\Sigma Y^\alpha, \hat{X}] \to [Y, \hat{X}] \to \varprojlim\, [Y^\alpha, \hat{X}] \to 0.$$

It remains to prove that $\beta \circ g \circ (\psi^q - 1) \simeq 0$ as infinite loop space maps. Consider the diagram of completions

$$\begin{array}{ccccc} JO(q) = JO(q)\hat{} & \xrightarrow{\beta\hat{}} & BO\hat{} & \xrightarrow{(\psi^q - 1)\hat{}} & BSO\hat{} \\ & & \downarrow{\scriptstyle g\hat{}} & & \downarrow{\scriptstyle g\hat{}} \\ & & BO\hat{} & \xrightarrow{(\psi^q - 1)\hat{}} & BSO\hat{} \end{array}$$

which commutes up to homotopy as a diagram of infinite loop maps because $\hat{g}$ is a linear combination of Adams operations ψ^k, k prime to p (see [7; 2.2]). Thus

$$[(\psi^q - 1)g\beta]\hat{} = (\psi^q - 1)\hat{}g\hat{}\beta\hat{} \simeq g\hat{}(\psi^q - 1)\hat{}\beta\hat{} \simeq 0;$$

however, since $JO(q)$ (and thus every delooping of $JO(q)$) has finite integral homology groups, completion is faithful. Thus $(\psi^q - 1) \circ g\beta \simeq 0$ as infinite loop space maps. □

Proof of Theorem A. According to (1.7) and Lemma 1.8,

$$\Phi - 1 = \tau \circ \Delta \circ (g\beta)$$

where $g\beta$ is a homotopy fibre of $\psi^q - 1$. However, β is some choice of homotopy fibre of $\psi^q - 1$ and so $g\beta$ is equally as good a choice. This completes the proof of Theorem A. □

Proof of Corollary C. The first assertion $\Phi^2 \simeq 1$ was established in [6; 2.13]. As for the second assertion,

$$\begin{aligned} KO^{-1}(JO(q)) &\approx KO^{-1}(BO(\mathbb{F}_q)) \approx \varprojlim KO^{-1}(BO_n(\mathbb{F}_q)) \\ &\approx \varprojlim RO\hat{}(0_n(\mathbb{F}_q)) = \mathbb{Z}/2\text{-module}. \end{aligned}$$

Thus $2(\Phi - 1) \simeq 0$, and similarly for each delooping $2B^n(\Phi - 1) \simeq 0$. Since $JO(q)$ has finite integral cohomology groups, these null-homotopies give a null-homotopy as maps of spectra. □

Proof of Corollary D. We analyze $\Phi - 1 = \tau\Delta B$ on homotopy groups. From fibre sequence (2) and Bott periodicity we have $\Delta_* = 0 : \pi_* BO \to \pi_* SO$ except in dimensions $8k$ where $\Delta_* : \mathbb{Z} \to \mathbb{Z}/2$ is reduction mod 2 and in dimensions $8k+1$ where $\Delta_* : \mathbb{Z}/2 \xrightarrow{\cong} \mathbb{Z}/2$. However from fibre sequence (1) of the introduction we have

$$\beta_* = 0 : \pi_{8k} JO(q) \to \pi_{8k} BO$$

and so we are reduced to dimensions $8k+1$. Again by sequence (2) we have

$$\beta_*(u_k) \neq 0, \qquad \beta_*(\eta_k) = 0. \quad □$$

2. Real representations and the proof of Theorem B

Let G be a finite group. The operation of assigning to each real representation of G its associated vector bundle over BG induces a homomorphism

$$\mathrm{RO}(G) \to [\mathrm{BG}, \mathrm{BO}]$$

which is compatible with Adams operations ψ^k. Atiyah and Segal [4] have proved that

$$\mathrm{RO}(G)^{\wedge} \xrightarrow{\approx} [\mathrm{BG}, \mathrm{BO}]$$

where completion is taken with respect to powers of the augmentation ideal $I(G)$ (i.e. completion of the $I(G)$-adic topology).

The fibration (1) of the introduction gives rise to an exact sequence

$$[\mathrm{BG}, \mathrm{SO}] \xrightarrow{\tau_*} [\mathrm{BG}, \mathrm{JO}(q)] \xrightarrow{\beta_*} [\mathrm{BG}, \mathrm{BO}] \xrightarrow{(\psi^q - 1)_*} [\mathrm{BG}, \mathrm{BSO}],$$

thus

$$[\mathrm{BG}, \mathrm{JO}(q)]/\mathrm{Im}\, \tau_* \xrightarrow[\approx]{\beta_*} [\mathrm{BG}, \mathrm{BO}]^{\psi^q} \xrightarrow[\approx]{} \mathrm{RO}(G)^{\wedge \psi^q}.$$

By the definition of coinvariants

$$[\mathrm{BG}, \mathrm{JO}(q)]_{\Phi} = [\mathrm{BG}, \mathrm{JO}(q)]/\mathrm{Im}(\Phi - 1)_*.$$

Thus the proof of Theorem B reduces to:

Proposition 2.1. *If some odd power of ψ^q acts idempotently on* RO(G), *then* $\mathrm{Im}\, \tau_* = \mathrm{Im}(\Phi - 1)_*$.

To prepare the proof of Proposition 2.1, let $\varrho : R(G) \to \mathrm{RO}(G)$ denote realification and set

$$\psi = \psi^q, \qquad A = \mathrm{RO}(G)/\varrho R(G), \qquad R = \mathrm{RO}(G).$$

It is straightforward to verify $(R^{\wedge})^{\psi} = (R^{\psi})^{\wedge}$, $(A_{\psi})^{\wedge} = (A^{\wedge})_{\psi}$ and so we can ignore the order of taking completion and (co-)invariants. Now extending fibration (1) of the introduction one term to the left, we have

$$\mathrm{SO} \xrightarrow{\Omega\psi - 1} \mathrm{SO} \xrightarrow{\tau} \mathrm{JO}(q) \xrightarrow{\beta} \mathrm{BO} \xrightarrow{\psi - 1} \mathrm{BO}. \tag{2.2}$$

Lemma 2.3. $A^{\wedge}_{\psi} \approx [\mathrm{BG}, \mathrm{SO}]_{\Omega\psi}$.

Proof. Atiyah–Segal [4] prove that $\mathrm{BO} \xrightarrow{\Delta} \mathrm{SO}$ induces an isomorphism

$$A^{\wedge} \xrightarrow{\approx} [\mathrm{BG}, \mathrm{SO}].$$

In order to prove that this map commutes with Adams operations we must show the homotopy commutativity of

$$\begin{array}{ccc} \mathrm{BO} & \xrightarrow{\Delta} & \mathrm{SO} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \Omega\psi} \\ \mathrm{BO} & \xrightarrow{\Delta} & \mathrm{SO} \end{array}$$

The operation $\Delta_*:\mathrm{KO}(\cdot)\to\mathrm{KO}^{-1}(\cdot)=\mathrm{KO}(S^1\wedge\cdot)$ is well known to be induces by smashing with the generator $\eta\in\pi_1\mathrm{BO}=\mathbb{Z}/2$. Thus we are reduced to showing the commutativity of

$$\begin{array}{ccc} \mathrm{KO}(\cdot) & \xrightarrow{\Delta_*} & \mathrm{KO}(S^1\wedge\cdot) \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ \mathrm{KO}(\cdot) & \xrightarrow{\Delta_*} & \mathrm{KO}(S^1\wedge\cdot) \end{array}$$

Since $\psi(\eta)=\eta$, this follows from the method of Adams proof of Corollary 5.3 of [1]. □

Let $\pi:R\to A$ denote projection and let $k:A^{\psi}\to A_{\psi}$ denote the natural map.

Lemma 2.4. *If some odd power of ψ^q is idempotent, then $k\pi$ induces an epimorphism*

$$\hat{k}:\hat{R}^{\psi}\to\hat{A}_{\psi}.$$

Proof. Let s be an odd integer for which $(\psi^q)^{2s}=(\psi^q)^s$. Suppose $a\in\hat{A}_{\psi}$ and $u\in\hat{R}$ is a preimage of a under the composite epimorphism

$$\hat{R}\to\hat{A}\to\hat{A}_{\psi}.$$

Consider the element

$$\bar{u}=(\psi^q)^s u+(\psi^q)^{s+1}u+\cdots+(\psi^q)^{2s-1}u.$$

Since $(\psi^q)^s$ is idempotent, $\psi^q\bar{u}=\bar{u}$. Hence $\bar{u}\in\hat{R}^{\psi}$. Moreover

$$\begin{aligned}\hat{k}\bar{u}&=(\psi^q)^s a+(\psi^q)^{s+1}a+\cdots+(\psi^q)^{2s-1}a\\&=a+a+\cdots+a\\&=sa=a,\end{aligned}$$

since A_{ψ} is a $\mathbb{Z}/2$ vector space. □

Proof of Proposition 2.1. Clearly $\mathrm{Im}(\Phi-1)_*\subset\mathrm{Im}\,\tau_*$ since $\Phi-1=\tau\Delta\beta$ by Theorem A. The opposite inclusion $\mathrm{Im}(\Phi-1)_*\supset\mathrm{Im}\,\tau_*$ follows from the commutative diagram (see (2.2))

$$\begin{array}{ccccc}
\hat{A}_\psi \approx & [BG, SO]_{\Omega\psi} & \xrightarrow{\tau_*} & & [BG, JO(q)] \\
\uparrow \hat{k} & \uparrow \Delta_* & & & \uparrow (\Phi - 1)_* \\
\hat{R}^\psi \approx & [BG, BO]^\psi & \xleftarrow{\beta_*} & & [BG, JO(q)]
\end{array}$$

where k is an epimorphism by Lemma 2.4. This completes the proof of Proposition 2.1 and thus Theorem B. □

We conclude with examples showing the necessity of the idempotence hypothesis in Theorem B. This topic will be explored more thoroughly in a forthcoming paper by the first author.

Proposition 2.5. *If G is a dihedral group of order 2^n, $n \geq 4$, then* $\hat{R}O(G)^{\psi^3} \neq [BG, JO(q)]_\Phi$.

Proof. It follows, from the arguments we used to prove Theorem B, that we need only show that

$$\hat{k} : \hat{R}^\psi \to \hat{A}_\psi$$

is not an epimorphism. Moreover, since A_ψ is a finite $\mathbb{Z}/2$ vector space, the completion $\hat{A}_\psi$ is attained in a finite number of stages. Hence it suffices to show that the composite

$$R^\psi \to \hat{R}^\psi \to \hat{A}_\psi$$

is not surjective.

Next we recall some facts about the representation theory of the dihedral group with 2^n elements (cf. Serre [12]). Denote by a, b the standard generators, subject to the relations

$$a^2 = b^{2^{n-1}} = 1, \qquad aba = b^{-1}.$$

Then the irreducible characters are given by Table 1 where $k = 1, 2, 3, \ldots, 2^{n-2} - 1$.

Table 1

	b^s	ab^s
χ_1	1	1
χ_2	1	-1
χ_3	$(-1)^s$	$(-1)^s$
χ_4	$(-1)^s$	$(-1)^{s+1}$
ϱ_k	$\left(2\cos\frac{sk}{2^{n-2}}\right)$	0

Note that all the irreducible characters are real, so $RO(G) = R(G)$. Hence $A = R \otimes \mathbb{Z}/2$. The characters multiply according to Table 2.

Table 2

	χ_1	χ_2	χ_3	χ_4	ϱ_l
χ_1	χ_1	χ_2	χ_3	χ_4	ϱ_l
χ_2	χ_2	χ_1	χ_4	χ_3	ϱ_l
χ_3	χ_3	χ_4	χ_1	χ_2	$\chi_{2^{n-2}-l}$
χ_4	χ_4	χ_3	χ_2	χ_1	$\varrho_{2^{n-2}-l}$
ϱ_k	ϱ_k	ϱ_k	$\varrho_{2^{n-2}-k}$	$\varrho_{2^{n-2}-k}$	$\varrho_{k+l}+\varrho_{k-l}$

To make sense of the entry in the lower right hand corner, one should note that ϱ_k depends only the residue class of $k \bmod 2^{n-1}$, that $\varrho_k=\varrho_{-k}$ and that

$$\varrho_0=\chi_1+\chi_2, \qquad \varrho_{2^{n-2}}=\chi_3+\chi_4.$$

The proof is completed by the following observations.

Observation 1. $\hat{A}=A$. Hence $\hat{A}_\psi=A_\psi$ and it suffices to show that the composite

$$R^\psi \to R \to A \to A_\psi$$

is not an epimorphism.

To see this one notes that $\hat{A}=A/\bigcap I^n$ where I is the image of the augmentation ideal under $R\to A$. Hence it suffices to show I is a nilpotent ideal. This is clear since I is generated by $\bar{\chi}_i+\bar{\chi}_1$, $i=2,3,4$ and $\bar{\varrho}_k$, $k=1,2,3,\dots,2^{n-2}-1$ (here $\bar{\chi}$ denotes the image of χ in A) and

$$(\bar{\chi}_i+\bar{\chi}_1)^2=\bar{\chi}_1+\bar{\chi}_1=0,$$
$$\bar{\varrho}_k^{2^{n-1}}=\bar{\varrho}_{2^{n-1}k}+\bar{\chi}_1+\bar{\chi}_2=\bar{\varrho}_0+\bar{\chi}_1+\bar{\chi}_2=0.$$

Observation 2. $\psi^3\chi_i=\chi_i$, $i=1,2,3,4$ and $\psi^3\varrho_k=\varrho_{3k}$. Hence R^ψ is free on the basis χ_i, $i=1,2,3,4$ and

$$\sum_{\substack{i\equiv k(\mathrm{mod}\ 2^{n-3})\\ 1\le i<2^{n-2}}} \varrho_i,$$

$k=1,2,4,\dots,2^{n-3}$. Similarly A_ψ has a basis $[\bar{\chi}_i]$, $i=1,2,3,4$ and $[\bar{\varrho}_k]$,. $k=1,2,4,\dots,2^{n-3}$. (Moreover ψ^3 has order 2^{n-3}, an even number.)

The first statement is obvious from the character formula $(\psi^3\alpha)(g)=\alpha(g^3)$. It follows that R^ψ is free on the traces of the orbits of $\{\chi_i,\varrho_k\}$ under the action of ψ^3. The second two statements now follow from the fact that 3 is a topological generator of the dyadic units.

Now under the map $R^\psi\to R\to A\to A_\psi$,

$$\sum_{\substack{i\equiv k(\mathrm{mod}\ 2^{n-3})\\ 1\le i<2^{n-2}}} \varrho_i$$

is sent to $(2^{n-3}/k)[\bar{\varrho}_k]$.

Hence the cokernel is generated by $[\varrho_k]$, $k=1,2,4,\ldots,2^{n-4}$, and the proof is complete. □

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